# Higher order hypergeometric Lauricella function and zero asymptotics of orthogonal polynomials

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A B S T R A C T

The asymptotic contracted measure of zeros of a large class of orthogonal polynomials is explicitly given in the form of a Lauricella function. The polynomials are defined by means of a three-term recurrence relation whose coefficients may be unbounded but vary regularly and have a different behaviour for even and odd indices. Subclasses of systems of orthogonal polynomials having their contracted measure of zeros of regular, uniform, Wigner, Weyl, Karamata and hypergeometric types are explicitly identified. Some illustrative examples are given.

*Keywords:*
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## 1. Introduction

It is an usual way of working in quantum many-body physics to transform the Hamiltonian operator of a physical system into an $N$-dimensional Jacobi matrix by means of the Lanczos algorithm or any of its numerous variants   It is also known that for a general $N \times N$ Jacobi matrix the characteristic polynomials of the principal submatrices form a set of orthogonal polynomials $\{P_n(x)\}_{n=1}^{N}$ which satisfy the recurrence relation

$$\begin{aligned} &P_n(x) = (x - a_n)\, P_{n-1}(x) - b_n^2\, P_{n-2}(x), \quad n = 1, 2, \ldots \\ &P_{-1}(x) = 0, \qquad P_0(x) = 1, \end{aligned} \tag{1}$$

where $a_n$ and $b_n$ are the Jacobi entries. It happens that the zeros of these orthogonal polynomials denote the energies of the levels of the physical system [3]. Here, following [4,5], we consider the class of systems of orthogonal polynomials defined by a recurrence relation of the previous type with coefficients satisfying the asymptotic conditions

$$\lim_{n\to\infty} \frac{a_{2n}}{\lambda_{2n}} = \alpha_1, \qquad \lim_{n\to\infty} \frac{b_{2n}}{\lambda_{2n}} = \beta_1, \qquad \lim_{n\to\infty} \frac{a_{2n+1}}{\lambda_{2n}} = \alpha_2, \qquad \lim_{n\to\infty} \frac{b_{2n+1}}{\lambda_{2n}} = \beta_2, \tag{2}$$

where $\lambda_n = g(n)$ is a regular varying function with exponent $\alpha \geq 0$. A regular varying function with exponent $\alpha$ can be written   as $g(x) = x^{\alpha} L(x)$ where $L : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function which satisfies $\lim_{x\to\infty} L(xt)/L(x) = 1$.

Associated with orthogonal polynomials of this type, with bounded ($\lambda_n = 1$ or $\alpha = 0$ and $L(x) = 1$) and unbounded ($\alpha > 0$) coefficients there exist a great variety of physical systems

## 2. Background and notations

For each polynomial $P_n(x)$, as defined by the recurrence (1), we consider its contracted and normalized zero counting measure

$$\rho_n := \frac{1}{n} \sum_{j=1}^{n} \delta\left(\frac{x_{j,n}}{\lambda_n}\right)$$

where $x_{j,n}$, $(j = 1, \ldots, n)$ are the zeros of $P_n(x)$, $\delta(x_{j,n}/\lambda_n)$ denotes the Dirac point mass at the scaled zero $x_{j,n}/\lambda_n$ and the scaling factor $\lambda_n$ is the $n$th element of the regular varying sequence such that the asymptotic behaviour shown in (2) holds true. It could be interesting to remark that when the family $\{P_n(x)\}_n$ satisfies a holonomic linear second order differential equation, the corresponding scaling factor can be obtained in terms of the coefficients characterizing such an equation

Our aim here is to express in terms of higher order hypergeometric Lauricella functions the corresponding asymptotic contracted measure of zeros for the sequence $\{P_n(x)\}_{n=1}^{N}$ to be denoted by $\rho$, i.e. a probability measure that satisfies

$$\lim_{n\to\infty} \int f \,\mathrm{d}\rho_n = \int f \,\mathrm{d}\rho$$

for every continuous function $f$ on $\mathbb{R}$ that vanishes at $\infty$. For this purpose, let us first introduce the following parameters (to be used throughout the paper)

$$\begin{aligned} \beta &= \left[\frac{1}{4}\,(\alpha_1-\alpha_2)^2 + (\beta_1+\beta_2)^2\right]^{1/2}, \qquad \gamma = \frac{1}{2}\,(\alpha_1+\alpha_2)\,, \\ \delta &= \left[\frac{1}{4}\,(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2\right]^{1/2}, \end{aligned} \tag{3}$$

where $\alpha_i$ and $\beta_i$ $(i = 1, 2)$ are the limits given in the above expressions (2). On doing this, the following well known result of van Assche [4] will play a essential role:

**Proposition 1** (*Theorem 4 (iii), [4]*). *Let $\{P_n(x)\}_{n=1}^{\infty}$ be an orthogonal polynomial sequence satisfying the recurrence* (1), *such that the $a_n$ and $b_n$ coefficients behave asymptotically as in* (2). *Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(x_{j,n}/\lambda_n) = \int_0^1 \int_{-\infty}^{+\infty} f(x)\mathrm{d}F(x-\gamma t^{\alpha};\, \delta t^{\alpha},\ \beta t^{\alpha})\mathrm{d}t, \tag{4}$$

*for every continuous function $f$. Here:*

(a) $\{\lambda_n\}_{n\in\mathbb{N}}$ *is a regularly varying sequence with exponent* $\alpha \geq 0$.
(b) $\beta, \gamma$ *and* $\delta$ *are the numbers defined in* (3).
(c) $F(x; u, v) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{|t|}{(v^2-t^2)^{1/2}(t^2-u^2)^{1/2}}\, \mathbb{I}_B(t)\mathrm{d}t$, *with* $B = [-v, -u] \bigcup [u, v]$ *being*

$$\mathbb{I}_B(t) = \begin{cases} 1 & \textit{if } t \in B \\ 0 & \textit{otherwise}. \end{cases}$$

On the other hand, the $F_D$-Lauricella function of order $n$ is defined by the series (cf. [11])

$$F_D^{(n)}\left[\begin{matrix} a\,;\, b_1\,,\ldots,\, b_n\,;\, c \\ x_1\,,\ldots,\, x_n \end{matrix}\right] = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a, m_1+\cdots+m_n)(b_1, m_1)\cdots(b_n, m_n)\, x_1^{m_1}\cdots x_n^{m_n}}{(c, m_1+\cdots+m_n)m_1!\ldots m_n!} \tag{5}$$

where $(a, j) = (a)_j$ stands for the Pochhammer symbol. Among others, this function satisfies the following reduction properties:

(a) If two variables coincide (e.g. $x_i = x_{i+1}$) then

$$F_D^{(n)}\left[\begin{matrix} a\,;\, b_1\,,\ldots,\, b_{i-1}\,,\, b_i\,,\, b_{i+1}\,,\ldots,\, b_n\,;\, c \\ x_1\,,\ldots,\, x_{i-1}\,,\, x_i\,,\ \ x_i\ \ ,\ldots,\, x_n \end{matrix}\right] = F_D^{(n-1)}\left[\begin{matrix} a\,;\, b_1\,,\ldots,\, b_{i-1}\,,\, b_i+b_{i+1}\,,\ldots,\, b_n\,;\, c \\ x_1\,,\ldots,\, x_{i-1}\,,\quad x_i\quad ,\ldots,\, x_n \end{matrix}\right]. \tag{6}$$

(b) In particular, the Lauricella function of two arguments ($F_D^{(2)}$) reduces to the so-called Appell hypergeometric function $F_1$:

$$F_D^{(2)}\left[\begin{matrix} a\,;\, b_1\,,\, b_2\,;\, c \\ x_1\,,\, x_2 \end{matrix}\right] = F_1[a, b_1, b_2; c; x_1, x_2], \tag{7}$$

where the notation of for $F_1$ is used.

With this background at hand, we are now in a position to show how the Lauricella $F_D^{(5)}$ function appears when trying to obtain the aforementioned asymptotic contracted measure of the zeros.

## 3. Main result: Appearance of the Lauricella $F_D^{(5)}$ function

The moments around the origin, $\mu_m$, $m = 1, 2, \ldots$, of the normalized asymptotic contracted measure of zeros are defined by

$$\mu_m = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{x_{j,n}}{\lambda_n}\right)^m = \int x^m \mathrm{d}\rho.$$

So, taking $f(x) = x^m$ in the previous Proposition 1 one can obtain an integral representation (see [13]) for all of them which reads as follows:

$$\mu_m = \frac{1}{\pi} \int_0^1 \mathrm{d}t \int_{-\infty}^{\infty} \frac{x^m |x - \gamma t^\alpha| \mathbb{I}_B(x - \gamma t^\alpha)}{\left[\beta^2 t^{2\alpha} - (x - \gamma t^\alpha)^2\right]^{1/2} \left[(x - \gamma t^\alpha)^2 - \delta^2 t^{2\alpha}\right]^{1/2}} \mathrm{d}x.$$

Now, we consider the case in which the exponent $\alpha$ of regular variation is positive (if $\alpha = 0$ no scaling of the variable is needed; see [14] for details of this case). Following the ideas of [13], it turns out that from these moments it is possible to construct the so-called characteristic function defined by

$$\Psi(s) = \sum_{m=0}^{\infty} \frac{(\mathrm{i}s)^m}{m!} \mu_m.$$

Its expression is [13]

$$\Psi(s) = \frac{1}{\pi} \int_C \frac{|y - \gamma| \mathrm{e}^{\mathrm{i}ys} {}_1F_1(1, 1 + 1/\alpha; -\mathrm{i}ys)}{[\beta^2 - (y - \gamma)^2]^{1/2} [(y - \gamma)^2 - \delta^2]^{1/2}} \mathrm{d}y \tag{8}$$

where $C = C_1 \bigcup C_2 = [\gamma - \beta, \gamma - \delta] \bigcup [\gamma + \delta, \gamma + \beta]$ and ${}_1F_1(1, 1 + 1/\alpha; -\mathrm{i}ys)$ stands for the confluent hypergeometric function ([12]).

Now, in the most general settings (i.e. when $\delta, \beta \in \mathbb{R}^+$ and $\gamma \in \mathbb{R}$ and $\beta \geq \delta$ (see (3)), the searched asymptotic contracted measure of zeros $\rho$ comes out from the Fourier transform of $\Psi(s)$ and can be written in terms of the Lauricella $F_D^{(5)}$ function. As illustration of this we show here the following:

**Theorem 2.** *Let $\{P_n(x)\}_{n=1}^{\infty}$ be an orthogonal polynomial sequence satisfying the recurrence* (1)*, such that the $a_n$ and $b_n$ coefficients behave asymptotically as in* (2)*. In addition, let $\beta$, $\gamma$ and $\delta$ be the numbers defined in* (3) *with $\beta > \delta > \gamma > 0$. Then, the asymptotic contracted density is*

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \begin{cases} \dfrac{|x|^{(1/\alpha)-1}}{\pi\alpha} I_1 & \text{if } x \in [\gamma - \beta, \gamma - \delta] \\ \dfrac{|x|^{(1/\alpha)-1}}{\pi\alpha} I_2 & \text{if } x \in [\gamma + \delta, \gamma + \beta] \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

*where*

$$I_1 = \left(\frac{2\beta(\beta - \gamma + x)}{\beta^2 - \delta^2}\right)^{1/2} (\beta - \gamma)^{-(1/\alpha)} \times F_D^{(5)} \left[ \begin{matrix} 1/2; & -1,\ 1/2, & 1/\alpha, & 1/2,\ 1/2; & 3/2 \\ \dfrac{\beta - \gamma + x}{\beta}, & \dfrac{\beta - \gamma + x}{2\beta}, & \dfrac{\beta - \gamma + x}{\beta - \gamma}, & \dfrac{\beta - \gamma + x}{\beta - \delta}, & \dfrac{\beta - \gamma + x}{\beta + \delta} \end{matrix} \right] \tag{10}$$

*and*

$$I_2 = \left(\frac{2\beta(\beta + \gamma - x)}{\beta^2 - \delta^2}\right)^{1/2} (\beta + \gamma)^{-(1/\alpha)} \times F_D^{(5)} \left[ \begin{matrix} 1/2; & -1,\ 1/2, & 1/\alpha, & 1/2,\ 1/2; & 3/2 \\ \dfrac{\beta + \gamma - x}{\beta}, & \dfrac{\beta + \gamma - x}{2\beta}, & \dfrac{\beta + \gamma - x}{\beta + \gamma}, & \dfrac{\beta + \gamma - x}{\beta - \delta}, & \dfrac{\beta + \gamma - x}{\beta + \delta} \end{matrix} \right]. \tag{11}$$

**Proof.** Using expression (8) of the characteristic function $\Psi$ and taking into account that Fourier's transform has the same value as density, that is to say,

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sx} \Psi(s) \mathrm{d}s,$$

the following integral representation is obtained

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=\frac{1}{\pi\alpha}\int_C\frac{|y-\gamma|(x/y)^{1/\alpha-1}\mathrm{d}y}{[\beta^2-(y-\gamma)^2]^{1/2}[(y-\gamma)^2-\delta^2]^{1/2}\,|y|}.$$

The assumption $\beta>\delta>\gamma>0$ allows us to conclude that $x$ and $y$ have the same sign and $|y|>|x|$, so

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=\frac{|x|^{(1/\alpha)-1}}{\pi\alpha}\int_C\frac{|y-\gamma|y^{-1/\alpha}\mathrm{d}y}{[\beta^2-(y-\gamma)^2]^{1/2}[(y-\gamma)^2-\delta^2]^{1/2}}.$$

Representing by $G$ the function inside the integral and having in mind our hypothesis we obtain $\gamma-\delta<0$ and $\gamma+\delta>0$, then the integral of $G$ in $C$ leads to $I_1^*=\int_{\gamma-\delta}^{x}G\mathrm{d}y$ and $I_2^*=\int_{x}^{\gamma+\beta}G\mathrm{d}y$, respectively.

On the other hand, the function $F_D^{(n)}$ defined in (5) admits the following integral representation (see [11])

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}\frac{(1-t)^{c-a-1}}{(1-x_1t)^{b_1}\,\ldots\,(1-x_nt)^{b_n}}\mathrm{d}t. \tag{12}$$

Then, after appropriate changes of variables, comparison of the above expression with (12) leads to (10) and (11). So, the assertion (9) follows and the Theorem 2 is proved. □

This general asymptotic density (9) of zeros considerably reduces when there exist some relations between the parameters $\delta$, $\beta$ and $\gamma$, i.e. when the coefficients of the recurrence relation (1) have special asymptotic behaviours.

## 4. Particular cases

Let us point out some of the particular relevant cases which are obtained from the reduction properties satisfied by the $F_D$ functions.

**Corollary 3.** *Let $\{P_n(x)\}_{n=1}^{\infty}$ be an orthogonal polynomial sequence satisfying the recurrence (1), such that the $a_n$ and $b_n$ coefficients behave asymptotically as in (2) and $\beta$, $\gamma$ and $\delta$ are the numbers defined in (3). Then*

(a) *If $\gamma=0$; $\beta,\ \delta\in\mathbb{R}^+$ (i.e. $\alpha_1=-\alpha_2$ in (2)), the asymptotic density of zeros reduces to*

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=\frac{|x|^{(1/\alpha)-1}}{\pi\alpha}\left(\frac{2(\beta-|x|)}{\beta^2-\delta^2}\right)^{1/2}\beta^{(\alpha-2)/2\alpha}F_D^{(4)}\left[\begin{matrix}1/2; & -1+1/\alpha,\,1/2, & 1/2,\,1/2; & 3/2\\ \dfrac{\beta-|x|}{\beta}, & \dfrac{\beta-|x|}{2\beta}, & \dfrac{\beta-|x|}{\beta-\delta}, & \dfrac{\beta-|x|}{\beta+\delta}\end{matrix}\right] \tag{13}$$

*if $x\in[-\beta,\ -\delta]\cup[\delta,\ \beta]$ and $\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=0$ otherwise, where $F_D^{(4)}$ is an $F_D$-Lauricella function of the fourth type* [11].

(b) *If $\delta=0$, $\beta\in\mathbb{R}^+$ and $\gamma\in\mathbb{R}$ (i.e. $\alpha_1=\alpha_2$; $\beta_1=\beta_2$ in (2)), the asymptotic contracted density becomes some Appell hypergeometric function. For instance, when $\gamma-\beta<0$ and $\gamma+\beta>0$ one has*

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=\begin{cases}\dfrac{|x|^{(1/\alpha)-1}}{\pi\alpha}I_3 & \text{if } x\in[\gamma-\beta,0]\\ \dfrac{|x|^{(1/\alpha)-1}}{\pi\alpha}I_4 & \text{if } x\in[0,\gamma+\beta]\\ 0 & \text{otherwise}\end{cases} \tag{14}$$

*where*

$$I_3=\left(\frac{2(\beta-\gamma+x)}{\beta}\right)^{1/2}(\beta-\gamma)^{-(1/\alpha)}F_1\left[1/2,1/2,1/\alpha;3/2;\frac{\beta-\gamma+x}{2\beta},\frac{\beta-\gamma+x}{\beta-\gamma}\right] \tag{15}$$

*and*

$$I_4=\left(\frac{2(\beta+\gamma-x)}{\beta}\right)^{1/2}(\beta+\gamma)^{-(1/\alpha)}F_1\left[1/2,1/2,1/\alpha;3/2;\frac{\beta+\gamma-x}{2\beta},\frac{\beta+\gamma-x}{\beta+\gamma}\right]. \tag{16}$$

*In particular, if $\gamma=0$, the asymptotic contracted density is*

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x}=\frac{|x|^{(1/\alpha)-1}}{\pi\alpha}(2(\beta-|x|))^{1/2}\,(\beta)^{-(\alpha+2)/2\alpha}F_1\left[1/2,1/2,1/\alpha;3/2;\frac{\beta-|x|}{2\beta},\frac{\beta-|x|}{\beta}\right] \tag{17}$$

*if $x\in[-\beta,\beta]$ and vanishes outside this interval.*

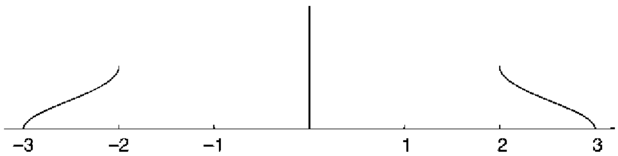


**Fig. 1.** $\alpha = 1, \beta = 3, \gamma = 0, \delta = 2.$

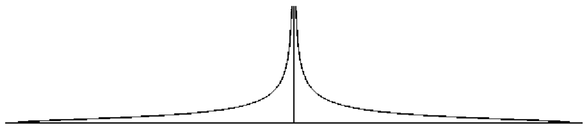

**Fig. 2.** $\alpha = 2, \beta = 1/2, \gamma = 0, \delta = 0.$

(c) *If* $\beta = \delta > 0$, $\gamma \in \mathbb{R}$, *the asymptotic contracted density is*

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \begin{cases} \dfrac{1}{2\alpha}\,\dfrac{1}{|\gamma-\beta^*|}\left(\dfrac{x}{\gamma-\beta^*}\right)^{(1/\alpha)-1} & \text{if } x \in [\gamma-\beta^*, 0] \\ \dfrac{1}{2\alpha}\,\dfrac{1}{|\gamma+\beta^*|}\left(\dfrac{x}{\gamma+\beta^*}\right)^{(1/\alpha)-1} & \text{if } x \in [0, \gamma+\beta^*] \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

*where* $\beta^* = [(1/4)\ (\alpha_1-\alpha_2)^2 + (\max(\beta_1,\beta_2))^2]^{1/2}$. *In particular, if* $\gamma = 0$,

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \begin{cases} \dfrac{1}{2\alpha\beta}\left(\dfrac{|x|}{\beta}\right)^{(1/\alpha)-1} & \text{if } x \in [-\beta, \beta] \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

**Proof.** (a) If $\gamma = 0$ the expressions (10) and (11) coincide. In addition to that, using the reduction property (6) with $n = 5$ and $x_1 = x_3$, we obtain the expression we were looking for (13). In Fig. 1 it is represented an example.

(b) If $\delta = 0$, then $C = [\gamma-\beta, \gamma-\delta]\bigcup[\gamma+\delta, \gamma+\beta] = [\gamma-\beta, \gamma+\beta]$. Considering that $0 \in C$ and using the reduction properties (6) (with $n = 5$ and $x_1 = x_4 = x_5$) and (7), we obtain that (10) and (11) conclude as expressions (15) and (16), respectively.

Moreover, if $\gamma = 0$ and using $|x|$, (15) and (16) coincide and replacing in (14) we obtain the density expression taken in (17). In Fig. 2 it is represented an example.

(c) Formula (4) when $\beta = \delta$ is transformed (see [4]) in

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{x_{n,i}}{\lambda_n}\right) = \frac{1}{2}\int_0^1 \left\{f[(\gamma+\beta^*)t^\alpha] + f[(\gamma-\beta^*)t^\alpha]\right\}\mathrm{d}t$$

where now $(\beta^*)^2 = (1/4)(\alpha_1-\alpha_2)^2 + \beta'^2$ with $\beta' = \max(\beta_1, \beta_2)$ and, $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ are the limits (2).

Taking $f(x) = x^m$ we obtain the following moment expressions

$$\mu_m = \frac{(\gamma+\beta^*)^m + (\gamma-\beta^*)^m}{2(m\alpha+1)}.$$

Moreover, using a similar argument as in the previous proof of Theorem 2, we obtain (18) for the contracted zero density. Last, we only need to put $\gamma = 0$ in (18) to obtain (19). So, Corollary 3 follows.

**Remark 4.** (1) In [15], other instances of the density (17) were shown to be the logarithmic function for $\alpha = 1$ and density function of polynomic type for $\alpha = \frac{1}{2(m+1)}$, $m = 0, 1, \ldots$; when $m = 0$ one has Wigner's semicircle law

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{2}{\pi\beta}\left(1 - \frac{x^2}{\beta^2}\right)^{1/2} \tag{20}$$

if $x \in [-\beta, \beta]$ and vanishes outside this interval.

(2) Notice that functions in (19) are called Karamata type functions. We would like to highlight the following particular cases:

(a) For $\alpha = 1$ we can see a rectangular (or uniform) density function centered at the origin.

(b) $\alpha = 2/3$ is the Weyl function.

*4.1. Some examples: The generalized Hermite polynomials*

The generalized Hermite polynomials $H_n^{(\mu)}(x)$ are orthogonal with respect to the measure $|x|^\mu \exp(-x^2)\mathrm{d}x\,(\mu > -1)$, and we denote by $K_n^{(\mu)}(x)$ its monic form. Then $K_n^{(\mu)}(x)$ satisfy a recurrence relation (1) where $a_n = 0$ and

$$b_n^2 = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+\mu}{2} & \text{if } n \text{ is odd.} \end{cases}$$

That is, according to the value of $\mu$ the following cases arise:

(a) If $\mu$ does not depend on the degree $n$ of the polynomial or $\mu = \mu(n)$ is a regular varying function with exponent smaller than 1, then $\alpha = 1/2$ and the parameter limits (2) are $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1/\sqrt{2}$. Therefore, the parameters (3) are $\beta = \sqrt{2}$, $\gamma = \delta = 0$ and the asymptotic contracted density is (17):

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{2|x|}{\pi}\sqrt{2(\sqrt{2}-|x|)}\,(\sqrt{2})^{\frac{-5}{2}}F_1\left[\frac{1}{2},\frac{1}{2},2;\frac{3}{2};\frac{\sqrt{2}-|x|}{2\sqrt{2}},\frac{\sqrt{2}-|x|}{\sqrt{2}}\right]$$

if $x \in [-\sqrt{2}, \sqrt{2}]$ and vanishes outside this interval. But this expression reduces to

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{\sqrt{2}}{\pi}\sqrt{1-\frac{x^2}{2}}$$

if $x \in [-\sqrt{2}, \sqrt{2}]$ and vanishes outside this interval and then Wigner's semicircle law behaviour (20) appears in this case.

(b) If $\mu = \mu(n)$ is a regular varying function with exponent 1, then $\alpha = 1/2$ and the parameter limits (2) are $\alpha_1 = \alpha_2 = 0$, $\beta_1 = 1/\sqrt{2}$ and $\beta_2 = 1$. Therefore, the parameters (3) are $\beta = 1 + 1/\sqrt{2}$, $\gamma = 0$ and $\delta = 1 - 1/\sqrt{2}$, then the asymptotic contracted density is (13):

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \frac{2|x|}{\pi}\sqrt{\frac{(1+1/\sqrt{2}-|x|)}{\sqrt{2}}}\,(1+1/\sqrt{2})^{-3/2}$$
$$\times F_D^{(4)}\left[\begin{matrix}1/2; & 1, 1/2, & 1/2, 1/2; & 3/2 \\ \dfrac{1+1/\sqrt{2}-|x|}{1+1/\sqrt{2}}, & \dfrac{1+1/\sqrt{2}-|x|}{2(1+1/\sqrt{2})}, & \dfrac{1+1/\sqrt{2}-|x|}{\sqrt{2}}, & \dfrac{1+1/\sqrt{2}-|x|}{2}\end{matrix}\right]$$

if $x \in [-1-1/\sqrt{2}, -1+1/\sqrt{2}\,] \cup [1-1/\sqrt{2}, 1+1/\sqrt{2}\,]$ and vanishes otherwise.

(c) If $\mu = \mu(n)$ is a regular varying function with exponent bigger than 1, then $\alpha = \nu/2$ and the parameter limits (2) are $\alpha_1 = \alpha_2 = \beta_1 = 0$ and $\beta_2 = 1/\sqrt{2}$. Therefore, the parameters $\beta^* = \delta = 1/\sqrt{2}$, and $\gamma = 0$, then the asymptotic contracted density is (19):

$$\frac{\mathrm{d}\rho(x)}{\mathrm{d}x} = \begin{cases} \dfrac{\sqrt{2}}{\nu}\left(\sqrt{2}|x|\right)^{(2/\nu)-1} & \text{if } x \in [-1/\sqrt{2}, 1/\sqrt{2}] \\ 0 & \text{otherwise .} \end{cases}$$

Notice that it is a Karamata type function such that when evaluated at $\nu = 2$ it becomes a rectangular (or uniform) density centered at the origin and for $\nu = 4/3$ is the Weyl function.

## 5. Concluding remarks

In conclusion, we have analytically determined in terms of hypergeometric Lauricella functions the asymptotic contracted measure of zeros of a large class of orthogonal polynomials. These are defined by a three-term recurrence relation, whose coefficients have a different limiting behaviour according to whether the indices are even or odd, as shown by (2). It has been obtained that in the most general case this contracted measure can be expressed in terms of an $F_D$-Lauricella function of fifth type, which simplifies a great deal when the asymptotic behaviour of the coefficients in the three-term recurrence relation takes a particular form.

Finally, it could be interesting to mention that the Lauricella functions obtained here represent asymptotic contracted measure of zeros and so, non-negativity of these functions in some intervals has been also proved as a byproduct.

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